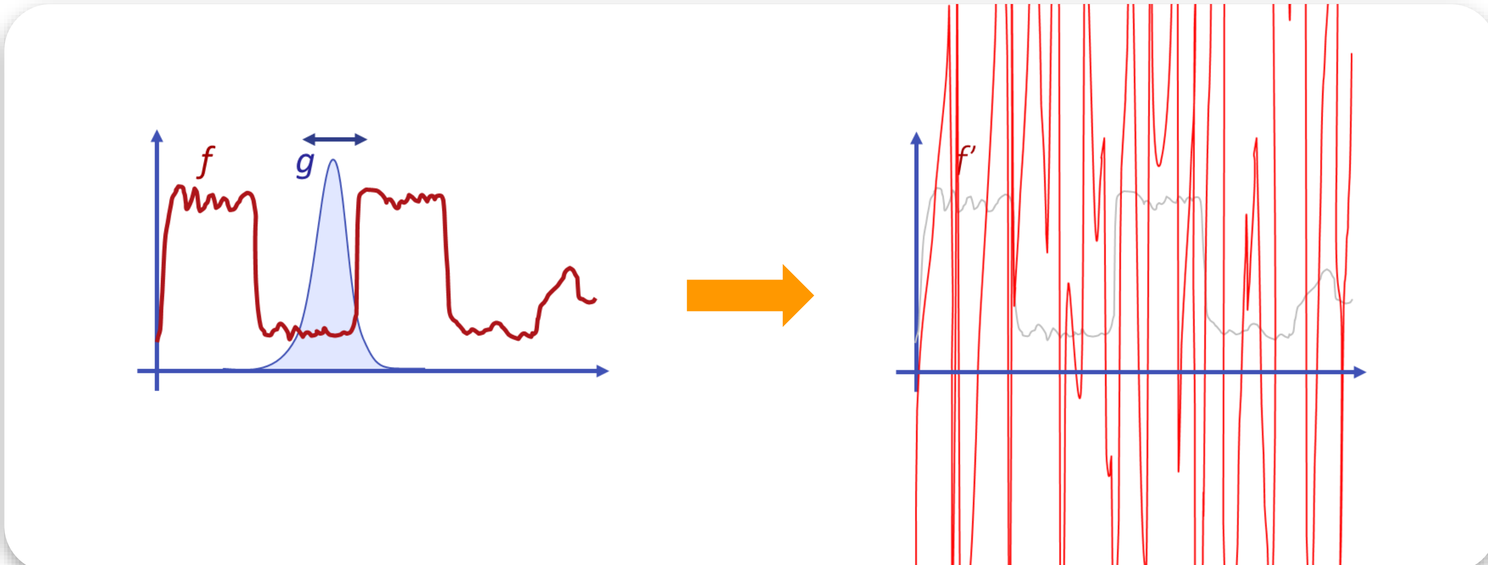
Modelling 1

SUMMER TERM 2020



LECTURE 10

Inverse Problems

Michael Wand · Institut für Informatik · Michael.Wand@uni-mainz.de

Linear Inverse Problems

Inverse Problems

Settings

- (Physical) process f
- Transforms original input $\mathbf{x}$ into output $\mathbf{b}$
- **Task:** recover $\mathbf{x}$ from $\mathbf{b}$

Examples:

- 3D structure from photographs
- Tomography: values from line integrals
- 3D geometry from a noisy 3D scan

Linear Inverse Problems

Simplification

- f is linear
- Finite dimensional input/output

$$f(\mathbf{x}) = \mathbf{b}$$

written as $\mathbf{A}_f \mathbf{x} = \mathbf{b}$

Then: Inversion of f is ill-posed, if...

- ...there is no solution.
- ...there is more than one solution.
- ...there is exactly one solution, but the SVD contains very small singular values.

Linear Inverse Problems

Simplification

- f is linear
- Finite dimensional input/output

$$f(\mathbf{x}) = \mathbf{b}$$

written as $\mathbf{A}_f \mathbf{x} = \mathbf{b}$

Then: Inversion of f is ill-posed, if...

- ...there is no solution.
- ...there is more than one solution.
- ...there is exactly one solution, but the SVD contains very small singular values.

remark:
formulation for continuous models (infinite-dim. spaces):
"the solution x depends continuously on b"

Example

Linear Operator

- Schauder Basis $b_1, b_2, b_3, \ldots$

$$f(x) = \sum_{k=1}^{\infty} \lambda_k b_k(x)$$

- Linear map $\lambda_k \to \frac{1}{k^2} \cdot \lambda_k$ is ill posed
- Inversion would be $\lambda_k \to k^2 \cdot \lambda_k$ (unbounded!)
- Example: Fourier basis
  Then this is the Laplace operator $\Delta = \partial_1^2 + \cdots + \partial_d^2$

Remark: General SVD

Linear Operator

- Orthogonal functions ("vectors")

$$u_1, u_2, u_3, \ldots : \mathbb{R} \to \mathbb{R}$$
$$v_1, v_2, v_3, \ldots : \mathbb{R} \to \mathbb{R}$$

- Scalars ("singular values")

$$\lambda_1, \lambda_2, \lambda_3, \ldots \in \mathbb{R}$$

- Linear map $L: (\mathbb{R} \to \mathbb{R}) \to (\mathbb{R} \to \mathbb{R})$, operates on functions $f: \mathbb{R} \to \mathbb{R}$

$$L(f) = \sum_{k=1}^{\infty} (\lambda_k \cdot \langle f, u_k \rangle) v_k$$

(exists under certain conditions, details beyond this course)

Finite Dim. Linear Inverse Problems

Simplifications

- f is linear
- Finite dimensional input/output

$$f(\mathbf{x}) = \mathbf{b}$$

written as $\mathbf{A}_f \mathbf{x} = \mathbf{b}$

Then: Inversion of f is ill-posed, if...

- ...there is no solution.
- ...there is more than one solution.
- ...there is exactly one solution, but the SVD contains very small singular values.

Ill-Posed Problems

Small singular values amplify error

- Inexact input
  - Measurement / numerical noise
- SVD

$$\mathbf{A}^{-1} = \mathbf{V}\,\mathbf{D}^{-1}\,\mathbf{U}^{\mathrm{T}}$$

no problem! (orthogonal)

decisive

no problem! (orthogonal)

Ill posed Problems

Ratio: Small singular values amplify errors

$$\mathbf{x} = \mathbf{A}^{-1}\mathbf{b} = (\mathbf{V}\mathbf{D}^{-1}\,\mathbf{U}^{\mathrm{T}})\mathbf{b}$$

- Example

$$\mathbf{D} = \begin{pmatrix} 2.5 & 0 & 0 & 0 \\ 0 & 1.1 & 0 & 0 \\ 0 & 0 & 0.9 & 0 \\ 0 & 0 & 0 & 0.000000001 \end{pmatrix}$$

- Noise amplified by 10^9
- Does *not* depend on *how* we invert the matrix.
- Condition number: $\sigma_{max} / \sigma_{min}$

Illustration of the Problem

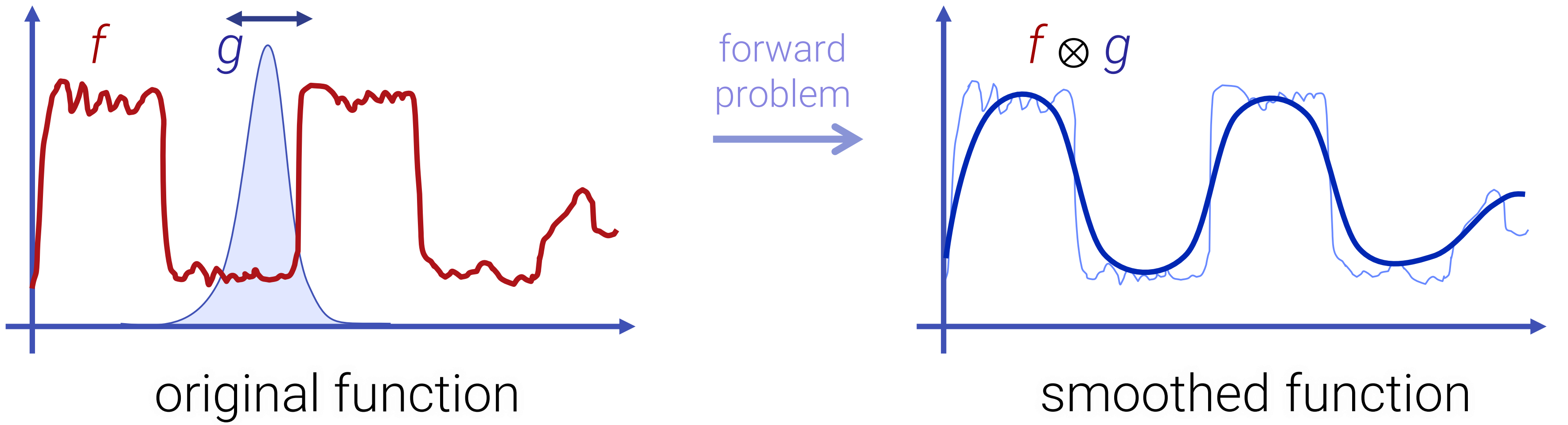


Illustration of the Problem

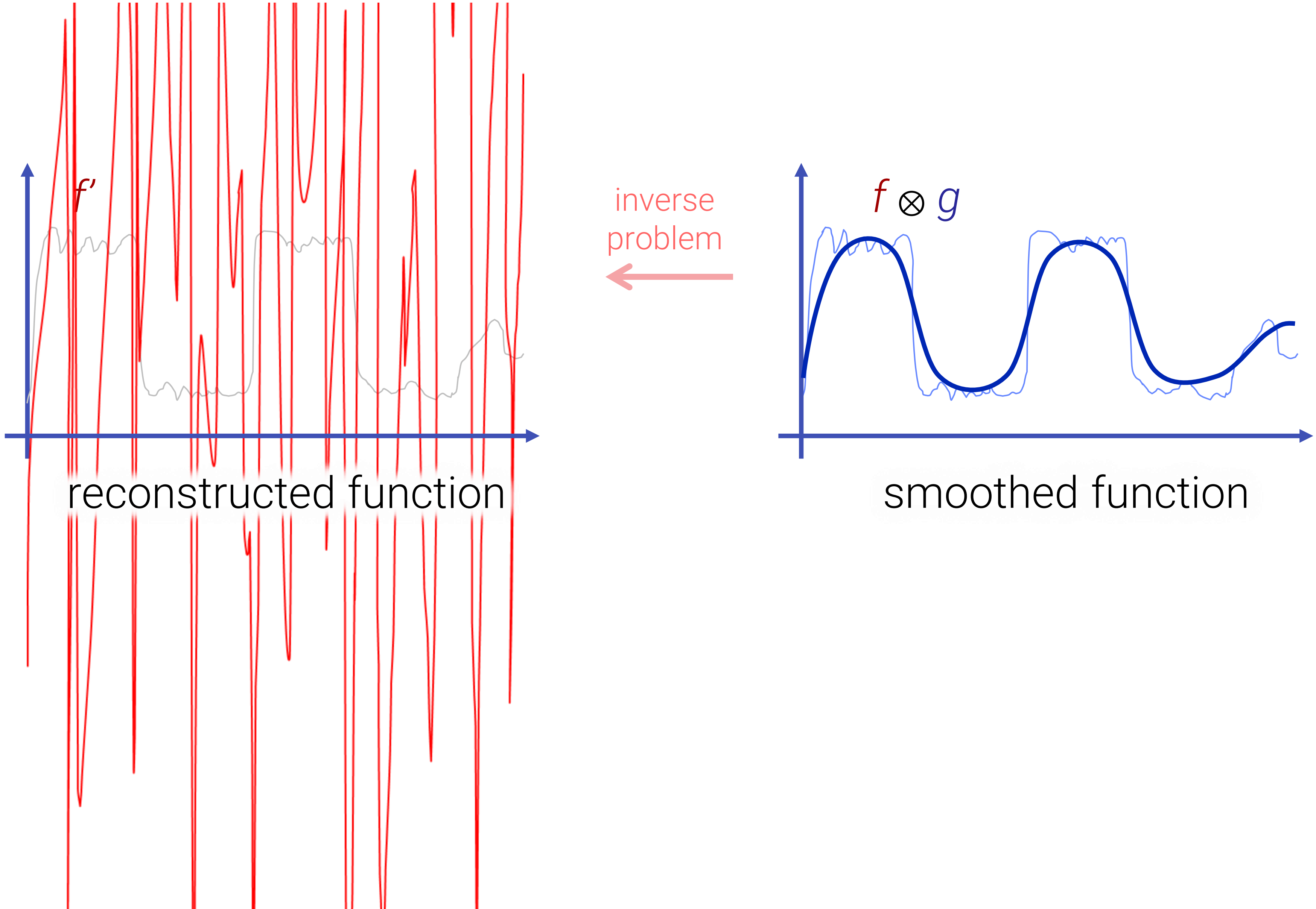
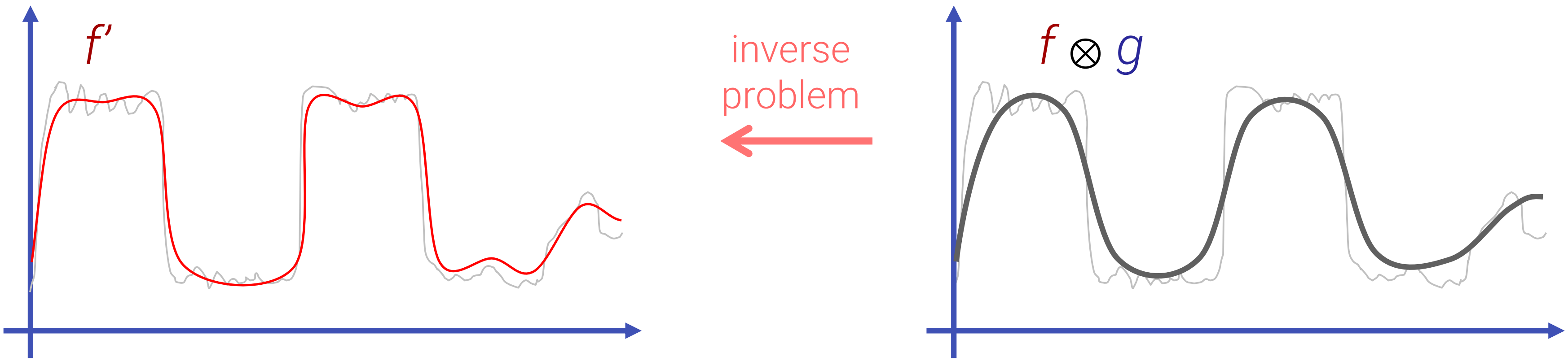


Illustration of the Problem



regularized
reconstructed function

smoothed function

Illustration of the Problem

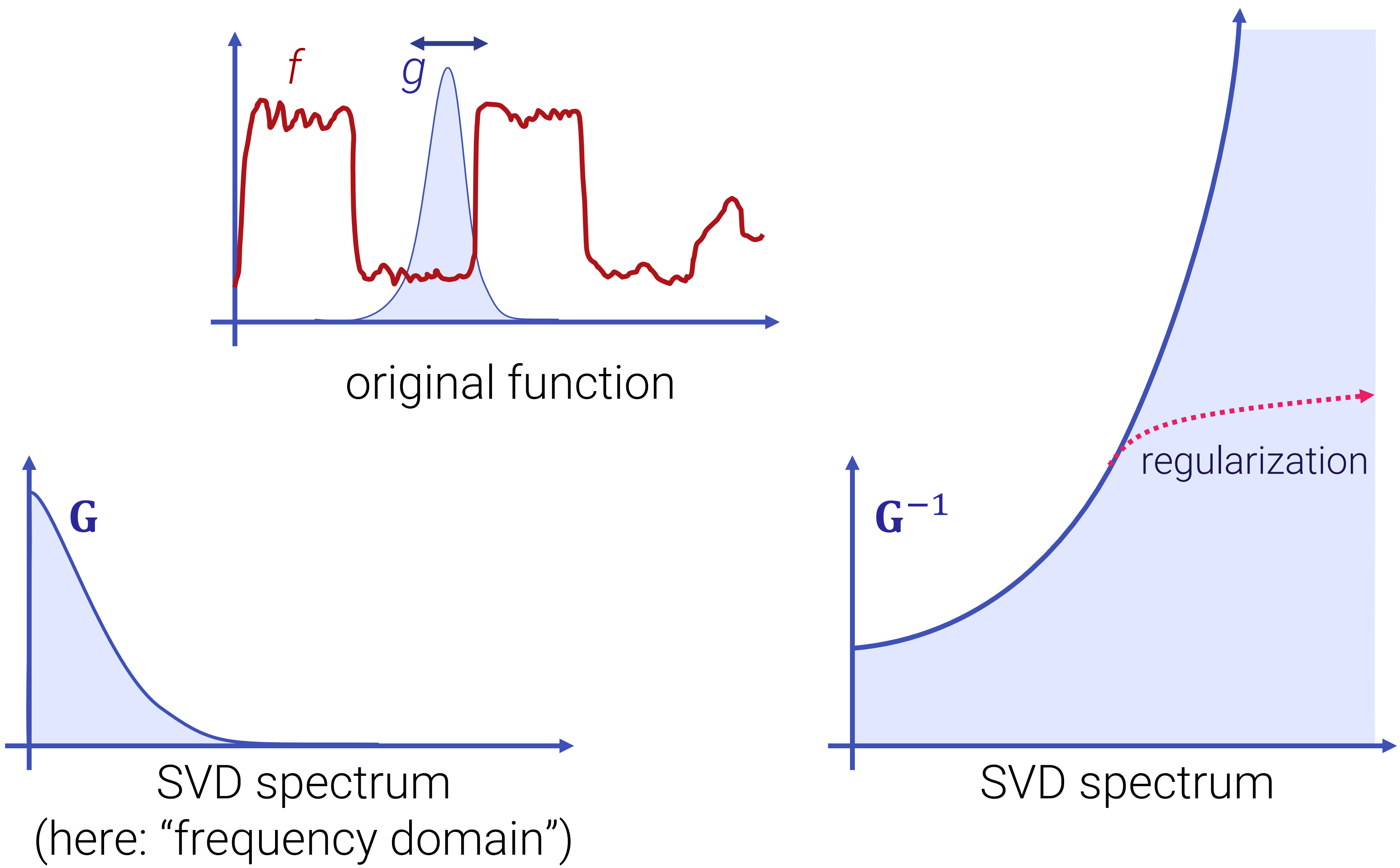
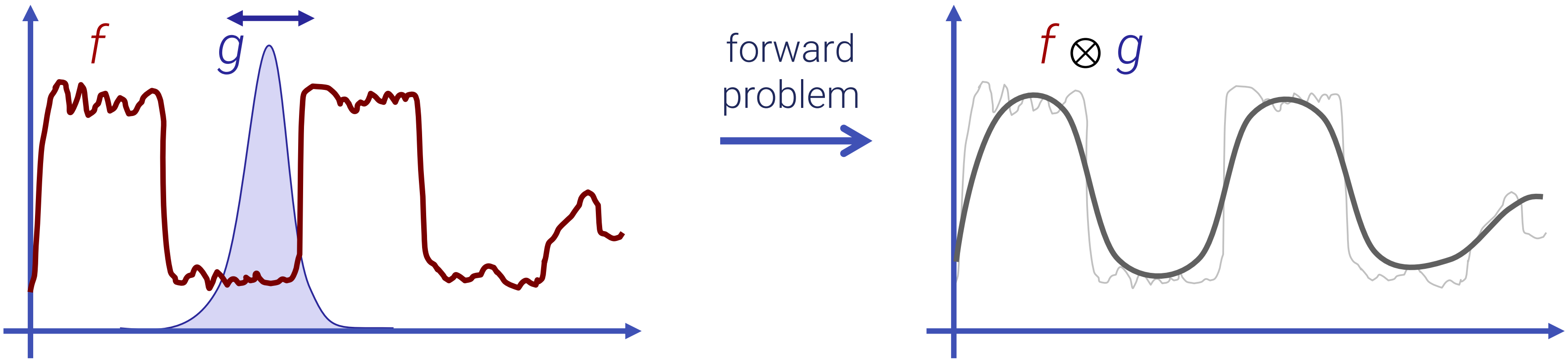


Illustration of the Problem

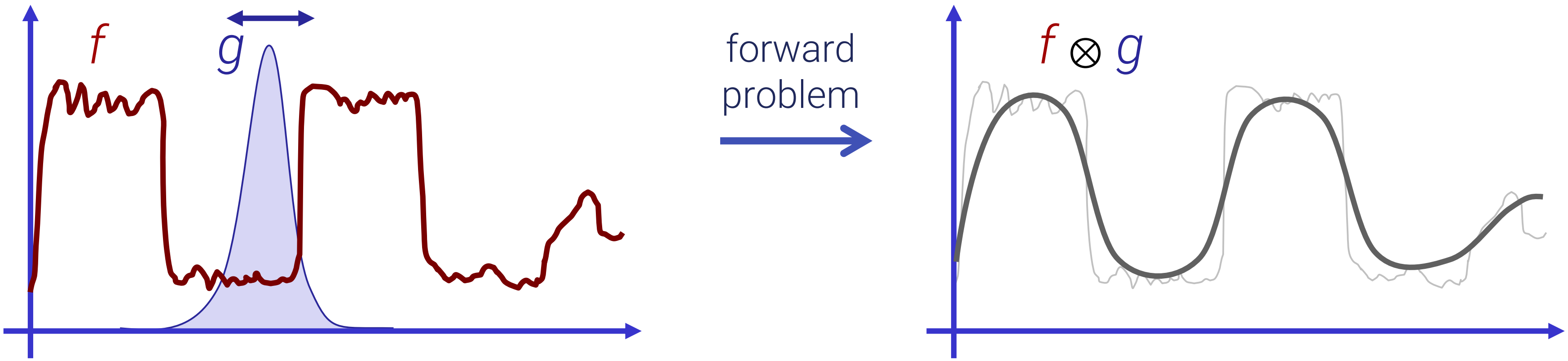


original function

smoothed function

$$
\begin{pmatrix} 1.2 \\ 1.5 \\ 0.3 \\ 0.4 \\ 1.6 \\ 0.2 \\ 0.3 \end{pmatrix} * \frac{1}{4} \begin{pmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 1.4 \\ 1.5 \\ 0.8 \\ 0.9 \\ 1.3 \\ 0.8 \\ 0.7 \end{pmatrix}
$$

Illustration of the Problem



original function

smoothed function

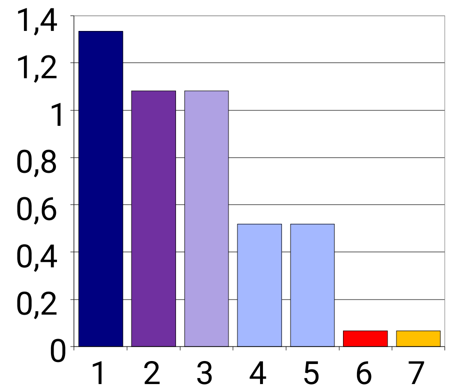
$$
\begin{pmatrix} 1.4 \\ 1.5 \\ 0.8 \\ 0.9 \\ 1.3 \\ 0.8 \\ 0.7 \end{pmatrix} * 4
\begin{pmatrix}
2 & 1 & 0 & 0 & 0 & 0 & 1 \\
1 & 2 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 2 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 2 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 2 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 2 & 1 \\
1 & 0 & 0 & 0 & 0 & 1 & 2
\end{pmatrix}^{-1}
=
\begin{pmatrix} 1.4 \\ 1.5 \\ 0.8 \\ 0.9 \\ 1.2 \\ 0.8 \\ 0.7 \end{pmatrix}
\qquad
\begin{pmatrix} 1.2 \\ 1.5 \\ 0.3 \\ 0.4 \\ 1.6 \\ 0.2 \\ 0.3 \end{pmatrix}
$$

solution (from 2 digits)   correct

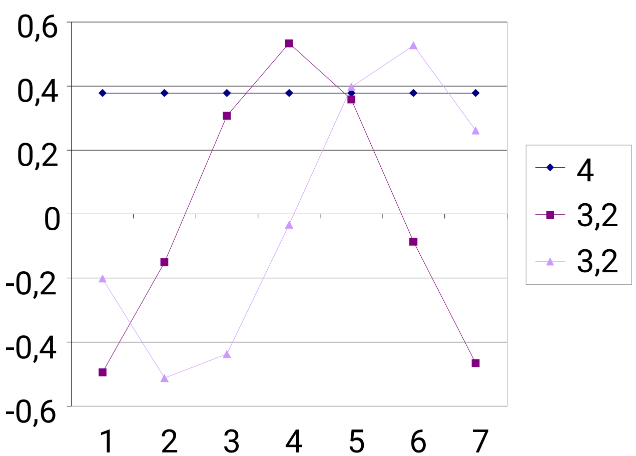
Analysis

$$\frac{1}{3}\begin{pmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 2 \end{pmatrix}$$

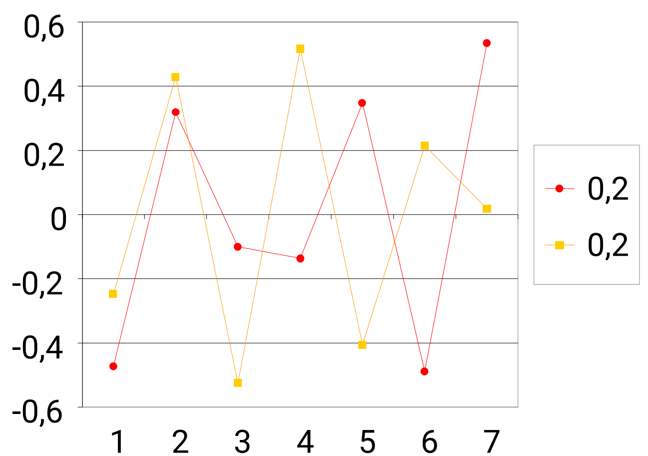
Matrix



Spectrum



Dominant Eigenvectors



Smallest Eigenvectors

Pseudo Inverse

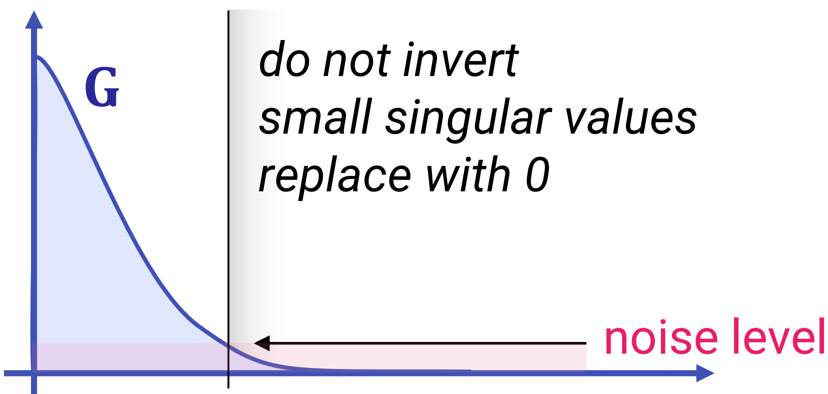
SVD Regularized Solve

- For full rank, square $\mathbf{A}$:

$$\mathbf{A} = \mathbf{U}\,\mathbf{D}\,\mathbf{V}^{\mathrm{T}}$$

$$\Rightarrow \mathbf{A}^{+} = (\mathbf{U}\,\mathbf{D}\,\mathbf{V}^{\mathrm{T}})^{-1} = (\mathbf{V}^{\mathrm{T}})^{-1}\,\widetilde{\mathbf{D}}^{-1}\,(\mathbf{U}^{-1}) = \mathbf{V}\,\widetilde{\mathbf{D}}^{-1}\,\mathbf{U}^{\mathrm{T}}$$

- Approximate inversion of $\mathbf{D}$

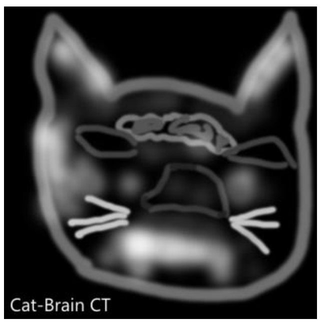


SVD spectrum
(here: “frequency domain”)

$$\mathbf{D} = \begin{pmatrix} 2.5 & 0 & 0 & 0 \\ 0 & 1.1 & 0 & 0 \\ 0 & 0 & 0.9 & 0 \\ 0 & 0 & 0 & 0.000000001 \end{pmatrix}$$

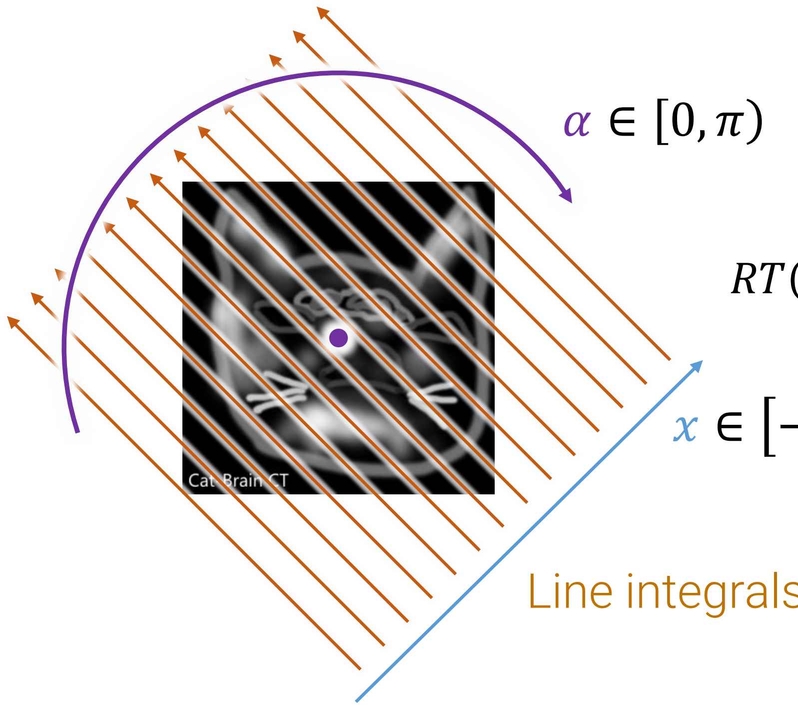
$$\widetilde{\mathbf{D}^{-1}} = \begin{pmatrix} 2.5^{-1} & 0 & 0 & 0 \\ 0 & 1.1^{-1} & 0 & 0 \\ 0 & 0 & 0.9^{-1} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Example: Tomography



Image

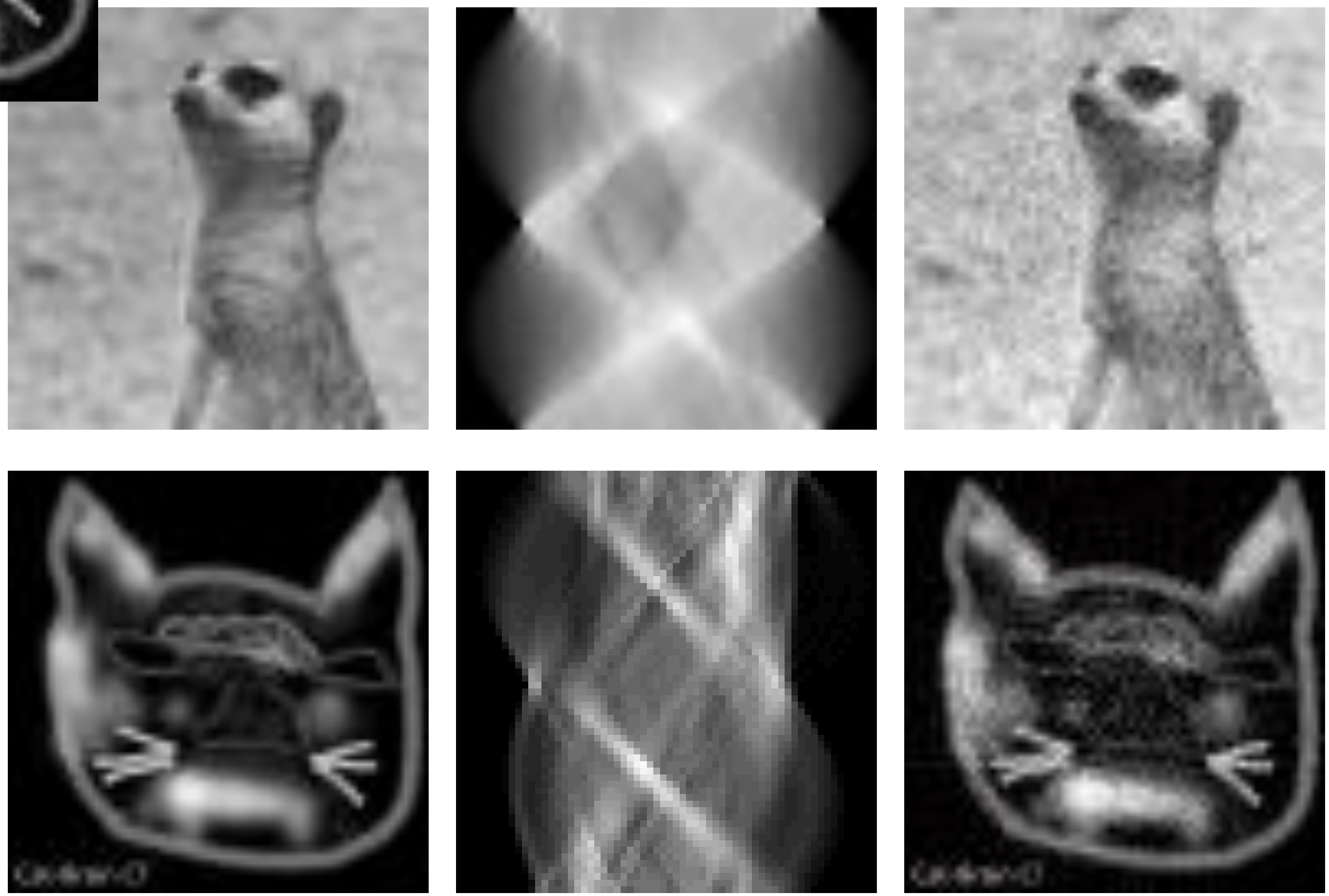
$f: [0,1]^2 \to \mathbb{R}$



Transformation

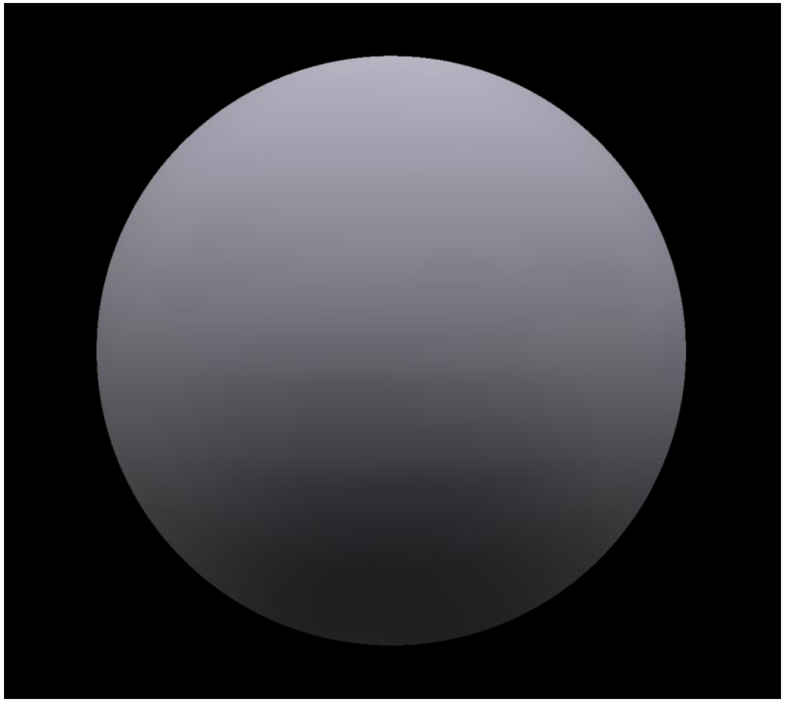
$RT(f): [-\sqrt{2}, \sqrt{2}\,] \times [0, \pi) \to \mathbb{R}$

Example: Tomography



Reflections: Spherical Convolution

Mirror Sphere
light transport operator
has full rank



Diffuse Sphere
light transport operator
has approx. rank 9
[Ramamoorthi et al. Siggraph 2001]